Completely regular semigroups and (Completely) $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups (a.k.a. U-superabundant semigroups): Similarities and Contrasts

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The setting

- Green's relations $\longrightarrow$ regular semigroup, simple semigroups, completely regular semigroups, inverse semigroups ...
- Generalizations to extended Green's relations $\mathcal{L}^*$, $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{L}}_E$, $\mathcal{L}^{(l)}$... (Fountain, Lawson, Shum, Pastijn...)
- **Objective**: study the analogs to completely regular (completely simple, Clifford) semigroups for relations $\widetilde{\mathcal{K}}_E$.
  - Emphazise on the similarities and differences.
  - Description as unary semigroups.
  - Application to regular semigroups.

Terminology

Terminology varies: abundant, semiabundant, weakly left abundant, left semiabundant, superabundant, U-semiabundant, weakly U-superabundant with C, weakly left ample, left E-ample, ...

Shum et al. proposed:

Definition

S is (A, σ)-abundant if each σ-class intersects A.

Green's extended relations

Extended Green's relations $\widetilde{\mathcal{L}}_E, \widetilde{\mathcal{R}}_E$ are based on (right, left) identities (El-Qallali'80, Lawson'90)

$$a\widetilde{\mathcal{L}}_E b \Longleftrightarrow \{(\forall e \in E)\ be = b \Leftrightarrow ae = a\};$$
$$a\widetilde{\mathcal{R}}_E b \Longleftrightarrow \{(\forall e \in E)\ eb = b \Leftrightarrow ea = a\}.$$

In general, $\widetilde{\mathcal{L}}_E$ is not a right congruence, $\widetilde{\mathcal{R}}_E$ is not a left congruence and the relations do not commute.

- $\widetilde{\mathcal{H}}_E = \widetilde{\mathcal{L}}_E \wedge \widetilde{\mathcal{R}}_E$;
- $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{L}}_E \vee \widetilde{\mathcal{R}}_E$;
- $\widetilde{\mathcal{J}}_E$ defined be equality of ideals.

The semigroups of this talk

- S is $(E, \widetilde{\mathcal{H}}_E)$-abundant if $(\forall a \in S, \exists e \in E)\ a\widetilde{\mathcal{H}}_E e$.
- S is completely $(E, \widetilde{\mathcal{H}}_E)$-abundant if it is $(E, \widetilde{\mathcal{H}}_E)$-abundant and $\widetilde{\mathcal{L}}_E, \widetilde{\mathcal{R}}_E$ are right and left congruences.
- S is completely E-simple if it is $(E, \widetilde{\mathcal{H}}_E)$-abundant and $\widetilde{\mathcal{D}}_E$-simple.
- S is an E-Clifford restriction semigroup if it is completely $(E, \widetilde{\mathcal{H}}_E)$-abundant with E central idempotents.

Other names exist (weakly U-superabundant, U-superabundant or weakly U-superabundant with C, completely $\widetilde{\mathcal{J}}_U$-simple).

Outline of the talk

1. Study as plain semigroups.
2. Study as unary semigroups.
3. Clifford and E-Clifford restriction semigroups.
4. Application: T-regular and T-dominated semigroups.

Study as plain semigroups

$(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups

Lemma

let S be a $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroup and $e, f \in E$. Then:

1. $e\widetilde{\mathcal{D}}_E f \Leftrightarrow e\mathcal{D}f$;
2. $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E$.

Proposition

let S be a $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroup, and let $e \in E$. Then

1. $\bigcup_{f \in E, f < e} fSf$ *is an ideal of* eSe;
2. $\widetilde{\mathcal{H}}_E(e) = eSe \backslash (\bigcup_{f \in E, f < e} fSf)$.

Union of monoids ? (1)

Proposition

Let S be a semigroup. Then S $(E, \widetilde{\mathcal{H}}_E)$-abundant **does not imply** *S is a disjoint union of monoids.*

Consider $S = \{0, a, 1_a, b, 1_b\}$ with multiplication table

	0	a	1_a	b	1_b
0	0	0	0	0	0
a	0	0	a	0	0
1_a	0	a	1_a	0	0
b	0	0	0	0	b
1_b	0	0	0	b	1_b

Pose $E = \{0, 1_a, 1_b\}$. Then S is $(E, \widetilde{\mathcal{H}}_E)$-abundant, but not a disjoint union of monoids (in particular, $\widetilde{\mathcal{H}}_E(1_a)$ is not a monoid).

$(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups and congruence

Theorem

Let S be a $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroup. Then the following statements are equivalent:

1. $\widetilde{\mathcal{L}}_E$ *and* $\widetilde{\mathcal{R}}_E$ *are right and left congruences;*
2. $\widetilde{\mathcal{D}}_E$ *is a semilattice congruence;*
3. $\widetilde{\mathcal{D}}_E$ *is a congruence.*

In this case, $\widetilde{\mathcal{J}}_E = \widetilde{\mathcal{D}}_E$, and each $\widetilde{\mathcal{H}}$-class is a monoid.

In particular, S is $(E, \widetilde{\mathcal{H}}_E)$-abundant and $\widetilde{\mathcal{D}}_E$-simple (completely E-simple) iff it is completely $(E, \widetilde{\mathcal{H}}_E)$-abundant and $\widetilde{\mathcal{J}}_E$-simple.

First structure theorems

Theorem

Let S be a completely $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroup. Then

1. *S is a disjoint union of monoids;*
2. *S is a semilattice of completely $E-$simple semigroups (see also Ren'2010).*

Converse is false.

Consider $S = \{0_a, 1_a\} \dot{\cup} \{0_b, 1_b\}$ with multiplication table

	0_a	1_a	0_b	1_b
0_a	0_a	0_a	0_b	0_b
1_a	0_a	1_a	0_b	0_b
0_b	0_a	0_a	0_b	0_b
1_b	0_a	0_a	0_b	1_b

Pose $E = \{1_a, 1_b\}$. Then S is not $(E, \widetilde{\mathcal{H}}_E)$-abundant.

Union of monoids ? (2)

Proposition

Let $S = \dot{\bigcup}_{e \in E} M_e$ *be a disjoint union of monoids such that*

$$(\forall a \in S, \forall e, f \in E)\ ae \in M_f \Rightarrow fe = f \quad \text{and } ea \in M_f \Rightarrow ef = f$$

Then S *is* $(E, \widetilde{\mathcal{H}}_E)$*-abundant.*
Conversely, any $(E, \widetilde{\mathcal{H}}_E)$*-abundant semigroup such that each* $\widetilde{\mathcal{H}}_E$*-class is a monoid is a union of monoids with this property.*

S is not completely $(E, \widetilde{\mathcal{H}}_E)$-abundant in general.
Let $S = \{f, e, d, a, a^2, \ldots\}$ such that $E = \{d, e, f\} = E(S)$ with $f \leq e \leq d$ and relations $ad = da = a$, $ae = ea = f$. It satisfies the assumptions of the proposition but $a \in \widetilde{\mathcal{L}}_E(d)$ whereas $f = ae \notin \widetilde{\mathcal{L}}_E(de = e)$.

Semillattice ?

Theorem

S is completely $(E, \widetilde{\mathcal{H}}_E)$-abundant if and only if it is a semilattice Y of $(E_\alpha, \widetilde{\mathcal{H}}_{E_\alpha})$-abundant, $\widetilde{\mathcal{D}}_{E_\alpha}$-simple semigroups such that:
$(\forall a \in S_\alpha, e \in E_\beta)$

$$f \in E_{\alpha\beta} \cap \widetilde{\mathcal{H}}_{E_{\alpha\beta}}(ae) \Rightarrow fe = f$$

and

$$f \in E_{\beta\alpha} \cap \widetilde{\mathcal{H}}_{E_{\beta\alpha}}(ea) \Rightarrow ef = f$$

The additional assumption is automatically satisfied for relation $\mathcal{H}$.

Completely E-simple semigroups

Theorem

The following statements are equivalent:

1. S *is* $(E, \widetilde{\mathcal{H}}_E)$*-abundant and* $\widetilde{\mathcal{D}}_E$*-simple;*
2. S *is completely* $(E, \widetilde{\mathcal{H}}_E)$*-abundant and* $\widetilde{\mathcal{J}}_E$*-simple;*
3. S *is* $(E, \widetilde{\mathcal{H}}_E)$*-abundant and the idempotents of* E *are primitive (within* E*).*

In particular,

$$\boxed{E = \{e \in E(S) | (\forall f \in E(S))\, ef = fe = e \Rightarrow e = f\} = Max}$$

set of maximal idempotents of S.

Completely E-simple semigroups

Proposition

S is completely E-simple iff it is the disjoint union of its local submonoids $eSe, e \in E$ and satisfies: $e, f \in E, efe = fe \Rightarrow fe \in E$ and $e, f \in E, efe = ef \Rightarrow ef \in E$.

Example: $\mathcal{C} = FinSet_n$

For $a \in Obj(\mathcal{C})$ choose $a \rightleftharpoons [n] = \{0, 1, \ldots, n-1\}$.

$(S = Mor(\mathcal{C}), \odot)$ with product

$$(a \to b) \odot (c \to d) = a \to b \to [n] \to c \to d$$

is completely E-simple, with $E = \{a \to [n] \to b | a, b \in Obj(\mathcal{C})\}$.

For $e = a \to [n] \to b$,

$$eSe = Mor(a, b) = \widetilde{\mathcal{H}}_E(e).$$

Rees-Suschkewitsch Theorem

Theorem

Let $\mathcal{M}(I, M, \Lambda, P)$ be a Rees matrix semigroup over a monoid with sandwich matrix with values in the group of units. Then $\mathcal{M}(I, M, \Lambda, P)$ is completely E-simple.
Conversely, any completely E-simple semigroup is isomorphic to a Rees matrix semigroup over a monoid with sandwich matrix with values in the group of units.

Corollary

S is completely E-simple iff $G_E = \dot{\bigcup}_{e \in E} G_e$ is a (completely simple) subsemigroup of S and $S = \dot{\bigcup}_{e \in E} eSe$.

example: $\mathcal{C} = FinSet_n$

Let $(S = Mor(\mathcal{C}), \odot)$ as before. Then

$$S \sim \mathcal{M}\left(obj(\mathcal{C}), Mor([n],[n]), obj(\mathcal{C}), (1)\right).$$

example:

$N = \langle a \rangle$ free monogenic semigroup, B nowhere commutative band. Pose $S = N \dot\cup B$ with product $a^n b = b a^n = a^n$, $b \in B, n \geq 0$. Then for any $e \in B$, $S = (N \dot\cup e) \bigcup_{f \in B \setminus \{e\}} \{f\}$ union of disjoint monoids with set of identities $E = B$.

$G_E = B$ completely simple but S **is not** $(B, \widetilde{\mathcal{H}}_B)$-abundant.

Assume a is $(B, \widetilde{\mathcal{H}}_B)$-related to $a_0 \in B$. As $fa = a = af$ then $fa_0 = a_0 f$ for any f, absurd.

Theorem (Hickey'10)

Let S be regular $J \subseteq S$ completely simple. $S = \dot{\bigcup}_{e \in E(J)} eSe$, if and only if $S \sim \mathcal{M}(I, T, \Lambda, P)$, T regular monoid and $P_{\lambda,i} \in T^{-1}$. In this case, $J \subseteq RP(S)$ and $E(J) = E(RP(S))$ where

$$RP(S) = \text{ Regularity Preserving elements of } S.$$

Corollary

Let S be a semigroup. Then the following statements are equivalent:

1. *S is a regular completely E-simple semigroup;*
2. *S is regular and $S = \dot{\bigcup}_{e \in E(J)} eSe$ for a completely simple subsemigroup J of S;*
3. *S is regular and $S = \dot{\bigcup}_{e \in E(RP(S))} eSe$.*

In this case, $J \subseteq RP(S) = \cup_{e \in E} G_e$ and $E = E(J) = E(RP(S))$.

Petrich Theorem

Petrich ('87) gives a construction of a completely regular semigroups from a given semilattice Y of Rees matrix semigroups. The same construction works in the setting of completely $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups (see also Yuan'14).
Extra ingredient needed: the structure maps $(\beta \leq \alpha)$

$$\phi_{\alpha,\beta} : S_\alpha = \mathcal{M}(I_\alpha, M_\alpha, \Lambda_\alpha, P_\alpha) \to M_\beta$$

must map $\mathcal{M}(I_\alpha, M_\alpha^{-1}, \Lambda_\alpha, P_\alpha)$ to M_β^{-1}.

Study as unary semigroups

The variety of $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups

We define a unary operation on $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups by:

$$\boxed{(\forall x \in S)\ x^+ \text{ is the unique element in } E \cap \widetilde{\mathcal{H}}_E(x)}$$

Conversely, for $(S, ., ^+)$ unary semigroup we pose

$$E = S^+ = \{x^+, x \in S\}$$

and

$$x\sigma^+ y \Leftrightarrow x^+ = y^+.$$

Let $(S, ., ^+)$ be a unary semigroup. We consider the following identities on $(S, ., ^+)$.

$$\begin{aligned}
x^+x &= x & (1)\\
xx^+ &= x & (2)\\
(xy^+)^+y^+ &= (xy^+)^+ & (3)\\
y^+(y^+x)^+ &= (y^+x)^+ & (4)\\
(x^+y)(xy)^+ &= x^+y & (5)\\
(yx)^+(yx^+) &= yx^+ & (6)\\
(xy)^+ &= (x^+y^+)^+ & (7)\\
(xy)^+x^+ &= x^+ & (8)\\
x^+(yx)^+ &= x^+ & (9)\\
x^+(xy)^+y^+ &= (xy)^+ & (10)
\end{aligned}$$

Theorem

1. $S^+\mathcal{A} = \mathcal{V}(1,2,3,4)$ *is the variety of unary* $(S^+, \widetilde{\mathcal{H}}_{S^+})$*-abundant semigroups;*
2. $\mathcal{C}S^+\mathcal{A} = \mathcal{V}(1,2,3,4,5,6)$ *is the subvariety of unary completely* $(S^+, \widetilde{\mathcal{H}}_{S^+})$*-abundant semigroups;*
3. $S^+\mathcal{G} = \mathcal{V}(1,2,3,4,7)$ *is the subvariety of unary completely* $(S^+, \widetilde{\mathcal{H}}_{S^+})$*-abundant,* $\widetilde{\mathcal{H}}_{S^+}$*-congruent semigroups (*S^+*-cryptogroups);*
4. $\mathcal{C}S^+\mathcal{S} = \mathcal{V}(1,2,8,9,10)$ *is the subvariety of unary completely* S^+*-simple semigroups.*

Moreover, $\mathcal{C}S^+\mathcal{S} \subseteq \mathcal{C}S^+\mathcal{G} \subseteq \mathcal{C}S^+\mathcal{A} \subseteq S^+\mathcal{A}$.

If $(S, ., ^+)$ belongs to any of these families, then $\sigma^+ = \widetilde{\mathcal{H}}_{S^+}$.

Clifford and E-Clifford restriction semigroups

Left restriction semigroup

A unary semigroup $(S, ., ^+)$ is a left restriction semigroup if

$$\begin{aligned} x^+x &= x \\ x^+y^+ &= y^+x^+ \ (S) \\ (x^+y)^+ &= x^+y^+ \ (LC) \\ xy^+ &= (xy)^+x \ (LA) \end{aligned}$$

In this case, $E = S^+ = \{x^+, x \in S\}$ is a semilattice and the unary operation is the identity on E.

Weakly left E-ample semigroup

Let S be a semigroup and $E \subseteq E(S)$ be a semilattice. Then S is weakly left E-ample if:

1. Every $\widetilde{\mathcal{R}}_E$-class $\widetilde{\mathcal{R}}_E(a)$ contains a (necessarily unique) idempotent a^+;
2. The relation $\widetilde{\mathcal{R}}_E$ is a left congruence;
3. The left ample condition $(\forall a \in S, \forall e \in E)\, ae = (ae)^+ a$ is satisfied.

Weakly left E-ample semigroups are precisely left restriction semigroups.

Clifford and E-Clifford restriction semigroup

Definition

A Clifford restriction semigroup $(S, ., ^+)$ *is a unary semigroup that satisfies the following identities:*

$$\begin{aligned} x^+x &= x \\ x^+y &= yx^+ \\ (xy)^{++} &= x^+y^+ \end{aligned}$$

Definition

S is a E-Clifford restriction semigroup if it is completely $(E, \widetilde{\mathcal{H}}_E)$-abundant with E central idempotents.

Theorem

Clifford restriction semigroup $\Leftrightarrow$ E-Clifford restriction semigroup.

Clifford restriction semigroup

Theorem

Let $(S, ., ^+)$ be a unary semigroup. Then the following statements are equivalent:

1. *S is a left restriction semigroup with $(xy)^+ = x^+y^+$;*
2. *S is a left restriction semigroup with $S^+ = \{x^+, x \in S\}$ semilattice of central idempotents;*
3. *S is a Clifford restriction semigroup;*
4. *$(S, ., ^+, ^+)$ is a restriction semigroup.*

E-Clifford restriction semigroup

Theorem

The following statements are equivalent:

1. S *is a* E*-Clifford restriction semigroup;*
2. S *is completely* $(E, \widetilde{\mathcal{H}}_E)$*-abundant and idempotents of* E *commute;*
3. S *is completely* $(E, \widetilde{\mathcal{H}}_E)$*-abundant and* $\widetilde{\mathcal{H}}_E = \widetilde{\mathcal{D}}_E$*;*
4. S *is* $(E, \widetilde{\mathcal{H}}_E)$*-abundant and* $\widetilde{\mathcal{H}}_E = \widetilde{\mathcal{D}}_E$ *is a congruence;*
5. S *is a semilattice* Y *of monoids* $\{F_\alpha, \alpha \in Y\}$*, with* $1_\alpha 1_\beta = 1_{\alpha\beta} (\forall \alpha, \beta \in Y)$*;*
6. S *is a strong semilattice* A *of monoids* $\{F_\alpha, \alpha \in Y\}$.

Also, S is a subdirect product of monoids but the converse does not hold.

Subdirect product

Proposition

Let S be a $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroup with E set of central idempotents of S. Then S is a subdirect product of the factors

$$\widetilde{\mathcal{H}}^0_E(e) = eSe / \left(\bigcup_{f \in E, f < e} fSf \right), \; e \in E$$

Let $M = \{0, n, 1\}, n^2 = 0$. The direct product $P = \{0\} \times M \times M$ is a commutative monoid and

$$S = \{(0,0,0); (0,n,0); (0,1,0); (0,0,n); (0,0,1)\}$$

is a subdirect product of $\{0\} \times M \times M$.

S is $(E(S), \widetilde{\mathcal{H}})$-abundant, but not completely $(E(S), \widetilde{\mathcal{H}})$-abundant.

Proper Clifford restriction semigroup

S E-Clifford restriction semigroup is proper if $\widetilde{\mathcal{H}}_E \cap \sigma = \iota$, where $\sigma = \{(a,b) \in S^2 | \exists e \in E, ea = eb\}$.
Let E be a semilattice, M a monoid, $OrdI(E)$ the set of order ideals of E. Let

$$I : (M, \leq_{\mathcal{J}}) \rightarrow (OrdI(E), \subseteq)$$

be a non-decreasing function (with $I(1) = E$).Then

$$\mathcal{M}(M,E,I) = \{(e,m) \in E \times M,\ e \in I(m)\}$$

with $(e,m)(f,n) = (ef, mn)$ and $(e,m)^+ = (e,1)$ is a proper Clifford restriction semigroup.

Theorem

S is a proper E-Clifford restriction semigroup if and only if it is isomorphic to a semigroup $\mathcal{M}(M,E,I)$.

Application: T-regular semigroups

Motivation

- Monoid M is a factorisable monoid (unit regular monoid) if

$$(\forall a \in S)\ a \in aM^{-1}a \quad (1)$$

- M inverse monoid is factorisable iff

$$(\forall a \in S, \exists x \in M^{-1})\ a\omega x \quad (2)$$

- Question: How to move from monoids to semigroups ? Can we get structure theorems ?
- Answer: To move from monoids to semigroups, replace M^{-1} by (some, all) maximal subgroups in (1) or (2).
- If $a\omega x$ with $x \in G_e = \mathcal{H}(e)$, then $ex = xe = x$.

T-regularity and domination

Definition

Let S be a regular semigroup, T a subset of S. $a \in S$ is T-regular (resp. T-dominated) if it admits an associate (resp. a majorant for the natural partial order) $x \in T$. S is T-regular (resp. T-dominated) if each element is T-regular (resp. T-dominated).

Lemma

Let $a \in S,\ x \in G_e, e \in E(S)$. Then

$$a\omega x \Longleftrightarrow ax^{\#}a = a,\ a \leq_{\mathcal{H}} x.$$

Structure Theorems

For $E \subseteq E(S)$, $G_E = \cup_{e \in E} G_e = \cup_{e \in E} \mathcal{H}(e)$.

Theorem

Let S be a semigroup. Then the following statements are equivalent:

1. *S is a completely E-simple, G_E-dominated semigroup;*
2. *S is a completely E-simple, G_E-regular semigroup;*
3. *S is isomorphic to a Rees matrix semigroup $\mathcal{M}(I, M, \Lambda, P)$ over a unit-regular monoid M with sandwich matrix with values in the group of units;*
4. *There exists a completely simple subsemigroup J of S, S is J-dominated and the local submonoids $eSe, e \in J$ are disjoint.*

Extends to completely $(E, \widetilde{\mathcal{H}}_E)$-abundant semigroups and E-Clifford restriction semigroups.